Numerical Quadrature Rules for Some Infinite Range Integrals

By Avram Sidi

Abstract. Recently the present author has given a new approach to numerical quadrature and derived new numerical quadrature formulas for finite range integrals with algebraic and/or logarithmic endpoint singularities. In the present work this approach is used to derive new numerical quadrature formulas for integrals of the form $\int_0^\infty x^\alpha e^{-x} f(x)\, dx$ and $\int_0^\infty x^\alpha E_p(x) f(x)\, dx$, where $E_p(x)$ is the exponential integral. It turns out the new rules are of interpolatory type, their abscissas are distinct and lie in the interval of integration and their weights, at least numerically, are positive. For fixed α the new integration rules have the same set of abscissas for all p. Finally, the new rules seem to be at least as efficient as the corresponding Gaussian quadrature formulas. As an extension of the above, numerical quadrature formulas for integrals of the form $\int_{-\infty}^{+\infty} |x|^\beta e^{-x^2} f(x)\, dx$ too are considered.

1. Introduction. In a recent paper, [8], the present author has presented a new approach to numerical quadrature for finite range integrals with algebraic and/or logarithmic endpoint singularities. In particular, the integrals dealt with are of the form

$$(1.1) \qquad I[f] = \int_0^1 w(x) f(x)\, dx,$$

where $w(x) = (1 - x)^\alpha x^\beta (-\log x)^\nu$, $\alpha + \nu > -1$, $\beta > -1$, and the numerical quadrature formulas are of the form

$$(1.2) \qquad I_k[f] = \sum_{i=1}^k A_{k,i} f(x_{k,i}), \qquad k = 1, 2, \dots.$$

These formulas are based on some rational approximations obtained by applying a modification of the nonlinear sequence transformations due to Levin [6], to the moment series of $w(x)$ and seem to have some very important advantages which we now summarize.

(1) The abscissas $x_{k,i}$ are all in the interval $[0, 1]$ and their weights turn out to be positive. For the weight functions $w(x)$ with $\alpha + \nu = 0, 1, 2, \dots$, that the abscissas are in $[0, 1]$ has been proved.

(2) The abscissas are the same for a large class of endpoint singularities; in particular, they are *independent of* β and depend solely on $\alpha + \nu$. Also if $\alpha + \nu$ is a small nonnegative integer like 0, 1, 2, the abscissas $x_{k,i}$ for the case $w(x) = 1$ can be taken to be the same for all these cases. In view of these remarks the weight functions $w(x) = 1$, $w(x) = x^\beta$, $w(x) = x^\beta(-\log x)$, $w(x) = x^\beta(-\log x)^2$, $w(x) = x^\beta(1 - x)^{1/2}(-\log x)^{-1/2}$, $w(x) = x^\beta(1 - x)^{1/2}(-\log x)^{1/2}$, etc., all have the same set of abscissas regardless of what $\beta > -1$ is.

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(3) The polynomials whose zeros are the abscissas $x_{k,i}$, are readily available in analytic form and the weights $A_{k,i}$ can be computed very easily once the abscissas have been determined.

(4) The new quadrature formulas have very strong convergence properties and are practically as efficient as the corresponding Gaussian formulas. For more details and numerical results the reader is referred to the paper of Sidi [8].

It turns out that the fruitful approach that has been used for the finite range integrals can be used for some infinite range integrals, among them some integrals that come up in radiation theory and involve the exponential integral as a weight function.

We shall now motivate the derivation of the new quadrature formulas for infinite range integrals in a manner analogous to that given in Sidi [8] and also exploit the motivation for introducing some of the notation to be used in the remainder of this work.

Let $f(z)$ be an analytic function in every finite subset of a simply connected open set G of the complex plane which contains the real infinite interval $[0, \infty)$. Let also Γ be a Jordan curve in G, containing the interval $[0, \infty)$ in its interior. Assume that $f(z) = O(z^{-a})$ as $z \to \infty$, $z \in G$, for some $a > 0$. Hence, by Cauchy's theorem, we have

$$(1.3) \qquad f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, dz,$$

whenever z_0 is in the interior of Γ and the integral is taken around Γ in the counterclockwise direction.

Define the infinite range integral $J[f]$ and its k-point numerical quadrature rule $J_k[f]$ by

$$(1.4) \qquad J[f] = \int_0^\infty w(x)f(x) \, dx, \qquad J_k[f] = \sum_{i=1}^k A_{k,i} f(x_{k,i}),$$

where the abscissas $x_{k,i}$ are distinct. Define also

$$(1.5) \qquad H(z) = \int_0^\infty \frac{w(x)}{z - x} \, dx, \qquad H_k(z) = \sum_{i=1}^k \frac{A_{k,i}}{z - x_{k,i}}.$$

The function $H(z)$ is analytic in the z-plane cut along the positive real axis $[0, \infty)$ and has the divergent asymptotic expansion

$$(1.6) \qquad H(z) \sim \sum_{n=1}^\infty \frac{\mu_n}{z^n} \quad \text{as } z \to \infty, z \notin [0, \infty),$$

where μ_n are the moments of $w(x)$,

$$(1.7) \qquad \mu_n = \int_0^\infty w(x) x^{n-1} \, dx, \qquad n = 1, 2, \ldots.$$

Let $E_k[f] = J[f] - J_k[f]$. Then with the help of (1.3), (1.4), and (1.5) it can be shown that, when all the $x_{k,i}$ are in the interior of Γ,

$$(1.8) \qquad |E_k[f]| \leqslant \frac{1}{2\pi} \int_\Gamma |H(z) - H_k(z)| \, |f(z)| \, |dz|,$$

provided the integral on the right-hand side exists. Now if this integral tends to zero as $k \to \infty$, then $E_k[f] \to 0$ as $k \to \infty$. As mentioned in [8], in order for the

integral in (1.8) and consequently for $E_k[f]$ to go to zero quickly as $k \to \infty$, we should demand that the sequence of the rational functions $H_k(z)$ tend to $H(z)$ uniformly and quickly in any region of the complex z-plane that is at a finite positive distance from the positive real axis.

If $H_k(z)$ is taken to be the $(k - 1/k)$ Padé approximant to (1.6), then $J_k[f]$ turns out to be the k-point Gaussian quadrature formula for $J[f]$. Furthermore, if $w(x) \geq 0$ on $[0, \infty)$, and the moments μ_n satisfy Carleman's condition, see Baker [2, Chapter 16] (this turns out to be the case for the $w(x)$ considered in this work), then as $k \to \infty$, $H_k(z) \to H(z)$ uniformly, in the sense of the previous paragraph.

In the present work we shall use again a modified version of the T-transformation of Levin to obtain another sequence of rational approximations $H_k(z)$, $k = 1, 2, 3, \ldots$, to $H(z)$ from the series (1.6) and use these $H_k(z)$ to derive new numerical integration rules for some useful weight functions like $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, and $w(x) = x^\alpha E_p(x)$, $\alpha > -1, p + \alpha > 0$, where $w(x) = E_1(x)$ and $w(x) = E_2(x)$ come up in radiation theory; see Chandrasekhar [3, Chapter 2]. We shall also show that these new rules have advantages similar to those for the singular finite range integrals of [8], which we have summarized at the beginning of this section. We note that these advantages are a direct outcome of the results of a recent work by the author, [10].

The Gaussian quadrature formulas for the weight function $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, have been widely tabulated. Gautschi [5] and Danloy [4] have considered the Gaussian quadrature rules for $w(x) = E_p(x), p > 0$, and have given some tables for the case $p = 1$.

2. Asymptotic Expansions for $H(z)$. As is explained in [8], see also [7], [9], whether or not one can apply the T-transformation successfully to the partial sums of the infinite series in (1.6), to obtain good approximations to $H(z)$, depends on whether $H(z)$ satisfies a relation of the form

$$(2.1) \qquad H(z) = \sum_{i=1}^{n-1} \frac{\mu_i}{z^i} + R_n f(n),$$

where R_n depend on the moments, and $f(y)$, as a function of the continuous variable y, has an asymptotic expansion of the form

$$(2.2) \qquad f(y) \sim \sum_{i=0}^{\infty} \frac{\beta_i}{y^i} \quad \text{as } y \to \infty,$$

and is an infinitely differentiable function of y up to and including $y = \infty$.

In a recent work, [10], the present author has shown that for $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, (2.1) is satisfied with

$$(2.3) \qquad \mu_i = (\alpha + i - 1)!, \qquad i = 1, 2, \ldots,$$

$$(2.4) \qquad R_n = \frac{(\alpha + n - 1)!}{nz^n},$$

and

$$(2.5) \qquad f(y) \sim \sum_{i=0}^{\infty} \frac{-ze^{-z}\delta_i(z)}{y^i} \quad \text{as } y \to \infty,$$

where

$$(2.6) \qquad \delta_i(z) = \frac{\partial^i}{\partial \xi^i} \left[e^{(1-\alpha)\xi} \exp(ze^\xi) \right] \Bigg|_{\xi=0}, \qquad i = 0, 1, \dots.$$

Furthermore, (2.1) holds for all $z \notin [0, \infty)$.

In [10] it is shown that also for $w(x) = x^\alpha E_p(x)$, $\alpha > -1$, $p + \alpha > 0$, where $E_p(x) = \int_1^\infty (e^{-xt}/t^p)\, dt$ is the exponential integral, whenever Re $z < 0$, (2.1) is again satisfied with

$$(2.7) \qquad \mu_i = \frac{(\alpha + i - 1)!}{p + \alpha + i - 1}, \qquad i = 1, 2, \dots,$$

and R_n exactly as given in (2.4), and $f(y)$ given by

$$(2.8) \qquad f(y) \sim \sum_{i=0}^\infty \frac{-z\varepsilon_i(z)}{y^i} \qquad \text{as } y \to \infty,$$

where

$$(2.9) \quad \varepsilon_i(z) = \frac{\partial^i}{\partial \xi^i} \left[e^{(1-\alpha)\xi} \exp(ze^\xi) \int_0^\xi d\sigma e^{(1-p)\sigma} \exp(-ze^\sigma) \right] \Bigg|_{\xi=0}, \qquad i = 0, 1, \dots.$$

Although the truth of (2.1) and (2.2) for this case has been shown only for Re $z < 0$, we believe that they hold also for Re $z \geqslant 0$.

For the details of the computations that lead to the results above, see [10].

3. Derivation of Numerical Quadrature Formulas and Some Properties.

We now take up the problem of derivation of the new numerical quadrature formulas $J_k[f]$ for the infinite range integrals $J[f]$ (see (1.4)) for the weight functions $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, and $w(x) = x^\alpha E_p(x)$, $\alpha > -1$, $p + \alpha > 0$. Let $A_0 = 0$ and $A_r = \sum_{i=1}^r \mu_i/z^i$, $r = 1, 2, \dots$, and $R_r = [(\alpha + r - 1)!/z^r]r^{-1}$, $r = 1, 2, \dots$. Applying the modified version of the T-transformation (see Eq. (2.2) in [8]), namely,

$$(3.1) \qquad T_{k,n} = \frac{\displaystyle\sum_{j=0}^k (-1)^j \binom{k}{j}(n + j)^{k-1} A_{n+j-1}/R_{n+j}}{\displaystyle\sum_{j=0}^k (-1)^j \binom{k}{j}(n + j)^{k-1}/R_{n+j}},$$

to the sequence above, and cancelling a factor of z from the numerator and the denominator, we obtain

$$(3.2) \qquad T_{k,n} = \frac{\displaystyle\sum_{j=0}^k (-1)^j \binom{k}{j}\frac{(n + j)^k}{(\alpha + n + j - 1)!} z^{n+j-1} A_{n+j-1}}{\displaystyle\sum_{j=0}^k (-1)^j \binom{k}{j}\frac{(n + j)^k}{(\alpha + n + j - 1)!} z^{n+j-1}}.$$

$T_{k,n}$ is an approximation to $H(z)$ in (1.5), obtained from the first $n + k - 1$ terms of its asymptotic series (1.6). It can easily be verified that $T_{k,n}$ is a rational function in z whose numerator $N_{k,n}(z)$ is a polynomial of degree $\leqslant n + k - 2$ and whose denominator $D_{k,n}(z)$ is z^{n-1} times another polynomial of degree exactly k, so that (degree of denominator) $\geqslant$ (degree of numerator) $+ 1$, a property that $H_k(z)$ must have (see (1.5)). $H_k(z)$ is also required to have simple poles, otherwise the numerical

quadrature formula contains derivatives of $f(x)$ as well as $f(x)$ itself, which is undesirable. Consequently, we must deal with the cases $n = 1, 2$ exclusively (the cases $n = 3, 4, \ldots$, give rise to a multiple pole at $z = 0$). Therefore, it seems that we could set $H_k(z) = T_{k,1}$ or $H_k(z) = T_{k-1,2}$ and use these $H_k(z)$ to derive numerical quadrature formulas $J_k[f]$ for the integral $J[f]$. For example, for $n = 1$, for which we give numerical results in this paper, we have

$$(3.3) \qquad H_k(z) = T_{k,1} = \frac{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(j+1)^k}{(\alpha+j)!} z^j A_j}{\displaystyle\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(j+1)^k}{(\alpha+j)!} z^j} = \frac{N_{k,1}(z)}{D_{k,1}(z)}.$$

Then the abscissas $x_{k,i}$ are the zeros of the denominator of $T_{k,1}$ (which are simple as we shall show below), and the weights $A_{k,i}$ are the residues of $T_{k,1}$ at the corresponding $x_{k,i}$, i.e.,

$$(3.4) \qquad A_{k,i} = \left. \frac{N_{k,1}(z)}{\dfrac{d}{dz} D_{k,1}(z)} \right|_{z=x_{k,i}}, \qquad i = 1, \ldots, k.$$

Remark 1. Since the denominators of $T_{k,1}$ and $T_{k-1,2}$ depend only on α and not on p, their zeros, therefore, the abscissas $x_{k,i}$ of our new numerical quadrature formulas, are independent of p. Consequently, we have the same set of abscissas for the weight functions $w_1(x) = x^\alpha e^{-x}$ and $w_2(x) = x^\alpha E_p(x)$, $\alpha > -1$, $p + \alpha > 0$. Furthermore, the polynomials that provide us with the abscissas are known analytically. This is not the case for the Gaussian formulas for the weight function $w_2(x)$.

Remark 2. The advantage of the new formulas mentioned in Remark 1 are a consequence of our flexible modification of the T-transformation of Levin. If Levin's t- (or u-) transformation is used to obtain the new numerical quadrature formulas, then, since $R_n = \mu_n/z^n$ (or $R_n = n\mu_n/z^n$), this advantage disappears altogether.

We demand of a good numerical quadrature formula that its abscissas be in the interval of integration. For the formulas obtained from $H_k(z) = T_{k,1}$ and $H_k(z) = T_{k-1,2}$ we can show that this holds true.

THEOREM 3.1. *Let*

$$(3.5) \qquad D_{k,n,m}(z) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(n+j)^m}{(\alpha+n+j-1)!} z^{n+j-1},$$

where m is a nonnegative integer. Then

$$(3.6) \qquad D_{k,n,m}(z) = \frac{k!}{(\alpha+n+k-1)!} \left(\frac{d}{dz} z \right)^m \left[z^{n-1} L_k^{(\alpha+n-1)}(z) \right],$$

where $L_k^{(s)}(z)$ is the generalized Laguerre polynomial of degree k. Furthermore, $D_{k,n,m}(z)$ has a zero of multiplicity $n - 1$ at $z = 0$ and k simple zeros in $(0, \infty)$ and between two simple zeros of $D_{k,n,m}(z)$ there is exactly one simple zero of $D_{k,n,m-1}(z)$, and the first positive zero of $D_{k,n,m}(z)$ is smaller than the first positive zero of $D_{k,n,m-1}(z)$.

Remark. Note that the case of interest in this work is $m = k$.

Proof. (3.6) follows from (3.5) by observing that

$$(3.7) \qquad D_{k,n,m}(z) = \left(\frac{d}{dz}z\right)^m \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{z^{n+j-1}}{(\alpha + n + j - 1)!}$$

and identifying the summation on the right-hand side of (3.7) as $C_{k,n}z^{n-1}L_k^{(\alpha+n-1)}(z)$, where $C_{k,n} = k!/(\alpha + n + k - 1)!$, see Abramowitz and Stegun [1, p. 775]. From (3.6) it follows that

$$(3.8) \qquad D_{k,n,m+1}(z) = \frac{d}{dz}\left[zD_{k,n,m}(z)\right], \qquad m = 0, 1, \ldots .$$

The rest of the theorem can now be proved by induction. Set $m = 0$. First $D_{k,n,0}(z) = C_{k,n}z^{n-1}L_k^{(\alpha+n-1)}(z)$, has a zero of multiplicity $n - 1$ at $z = 0$ and k simple zeros $x_i^{(0)}$, $i = 1, \ldots, k$, in $(0, \infty)$ which we order as $0 < x_1^{(0)} < x_2^{(0)} < \cdots < x_k^{(0)}$, namely those of $L_k^{(\alpha+n-1)}(z)$. Therefore, $D_{k,n,1}(z) = (d/dz)[zD_{k,n,0}(z)]$, by Rolle's theorem, has a zero of multiplicity $n - 1$ at $z = 0$ and k simple zeros $x_i^{(1)}$, $i = 1, \ldots, k$, such that $0 < x_1^{(1)} < x_1^{(0)} < x_2^{(1)} < x_2^{(0)} < \cdots < x_k^{(1)} < x_k^{(0)}$. The rest of the proof for general m is now obvious.

By letting $m = k$ in Theorem 3.1 we have the following result.

COROLLARY. *$T_{k,1}$ has k simple poles in $(0, \infty)$ and $T_{k-1,2}$ has a simple pole at $z = 0$ and $k - 1$ simple poles in $(0, \infty)$; i.e., the abscissas of the numerical quadrature formulas obtained by setting $H_k(z) = T_{k,1}$ (or $H_k(z) = T_{k-1,2}$) are distinct, real and in $(0, \infty)$ (or in $[0, \infty)$ with $z = 0$ being one of them).*

THEOREM 3.2. *Let $\sigma_j, j = 0, 1, 2, \ldots,$ be constants independent of k, and define*

$$(3.9) \qquad G_{k,n,m}(z) = \sum_{j=0}^{k} \binom{k}{j}\sigma_j(n + j)^m z^{n+j-1} = \left(\frac{d}{dz}z\right)^m \sum_{j=0}^{k} \binom{k}{j}\sigma_j z^{n+j-1}.$$

Then

$$(3.10) \qquad G_{k,n,m}(z) = (n + k)G_{k,n,m-1}(z) - kG_{k-1,n,m-1}(z).$$

Proof. We can express (3.9) in the form

$$(3.11) \qquad \begin{aligned} G_{k,n,m}(z) &= \left(\frac{d}{dz}z\right)^{m-1}\frac{d}{dz}\sum_{j=0}^{k}\binom{k}{j}\sigma_j z^{n+j} \\ &= \left(\frac{d}{dz}z\right)^{m-1}\sum_{j=0}^{k}\binom{k}{j}\sigma_j[(n + k) - (k - j)]z^{n+j-1}. \end{aligned}$$

The result in (3.10) now follows by noting that $\binom{k}{j}(k - j) = k\binom{k-1}{j}$ for $j = 0, \ldots, k - 1$ and 0 for $j = k$.

COROLLARY. *The polynomials $D_{k,n,m}(z)$ satisfy*

$$(3.12) \qquad D_{k,n,m}(z) = (n + k)D_{k,n,m-1}(z) - kD_{k-1,n,m-1}(z).$$

Proof. (3.12) follows from the fact that $D_{k,n,m}(z)$ are polynomials of the form of $G_{k,n,m}(z)$ in Theorem 3.2 above with $\sigma_j = (-1)^j/(\alpha + n + j - 1)!$, $j = 0, 1, \ldots .$

Making use of Theorem 3.2, we can prove that the abscissas of our new numerical quadrature formulas have the interlacing property like those of the Gaussian quadrature formulas.

THEOREM 3.3. *Let $x_1^{k,m} < x_2^{k,m} < \cdots < x_k^{k,m}$ be the simple zeros of $D_{k,n,m}(z)$ in $(0, \infty)$ (see Theorem 3.1). Then between two consecutive such zeros of $D_{k,n,m}(z)$ there is exactly one simple zero of $D_{k-1,n,m-1}(z)$, $m \geqslant 2$.*

Proof. If we let $z = x_j^{k,m}$ and $z = x_{j+1}^{k,m}$, $1 \leqslant j \leqslant k - 1$ in (3.12) and use the fact that $D_{k,n,m}(x_i^{k,m}) = 0$, $i = j, j + 1$, we obtain

$$(3.13) \qquad kD_{k-1,n,m-1}(x_i^{k,m}) = (n + k)D_{k,n,m-1}(x_i^{k,m}), \qquad i = j, j + 1.$$

In Theorem 3.1 we showed that between two simple zeros of $D_{k,n,m}(z)$ in $(0, \infty)$ there is exactly one simple zero of $D_{k,n,m-1}(z)$. Therefore, $D_{k,n,m-1}(x_i^{k,m}) \neq 0$, $i = 1, 2, \ldots, k$, furthermore $D_{k,n,m-1}(x_j^{k,m})$ and $D_{k,n,m-1}(x_{j+1}^{k,m})$ have opposite signs. This together with (3.13) implies that $D_{k-1,n,m-1}(x_j^{k,m})$ and $D_{k-1,n,m-1}(x_{j+1}^{k,m})$ have opposite signs too. Therefore, $D_{k-1,n,m-1}(z)$ must vanish at least once in $(x_j^{k,m}, x_{j+1}^{k,m})$. But since the number of simple zeros of $D_{k-1,n,m-1}(z)$ on $(0, \infty)$ is $k - 1$, there can be at most one zero in $(x_j^{k,m}, x_{j+1}^{k,m})$. This completes the proof of the theorem.

COROLLARY. *The positive abscissas of the new numerical quadrature formulas obtained from the rational functions $T_{k,n}$ and $T_{k-1,n}$ ($n = 1, 2$) interlace.*

Although we have not proved the positiveness of the weights $A_{k,i}$ of our new numerical quadrature formulas, numerical results indicate that they are. This, as is known, is an important property that numerical quadrature formulas are required to have.

Another property of our new formulas is that they are of interpolatory type, see [8], as the following result states.

THEOREM 3.4. *Let $J_k[f]$ in (1.4) be the numerical quadrature formula associated with $H_k(z) = T_{k,1}$ or $H_k(z) = T_{k-1,2}$. Then*

$$(3.14) \qquad\qquad J_k[x^i] = J[x^i], \qquad i = 0, 1, \ldots, k - 1,$$

both for $w_1(x) = x^\alpha e^{-x}$ and $w_2(x) = x^\alpha E_p(x)$, $\alpha > -1, p + \alpha > 0$, and

$$(3.15) \qquad\qquad J_k[x^i] = J[x^i], \qquad i = 0, 1, \ldots, k,$$

for $w_2(x)$ whenever $p + \alpha = 1$.

Proof. The same as that of Theorem 4.1 and the remark following it in [8].

Remark 3. Theorems 3.1 and 3.3 and their corollaries are similar in nature to the corresponding theorems in [8]. The result of Theorem 3.2, however, is new and contains Eq. (4.14) in [8] as a special case.

4. Numerical Results. In this section we shall give some numerical results obtained by taking $H_k(z) = T_{k,1}$. The results for $H_k(z) = T_{k-1,2}$ are about the same as those for $H_k(z) = T_{k,1}$ (they both are k-point formulas) and will not be given here.

TABLE 4.1a

The abscissas $x_{k,i}$ for the new numerical quadrature formulas $\int_0^\infty w(x)f(x)\,dx \approx$
$\sum_{i=1}^k A_{k,i}(x_{k,i})$ where $w(x) = x^\alpha e^{-x}$ or $w(x) = x^\alpha E_p(x)$, $p > 0$. These $x_{k,i}$ are the roots of the
polynomial equation $\sum_{j=0}^k \lambda_j z^j = 0$ with $\lambda_j = (-1)^j\binom{k}{j}(j + 1)^k/j!$, $j = 0, 1, \ldots, k$. The
weights $A_{k,i}$ can be computed from (4.2).

K = 2

0.135296668541636873437271 5Q+00
0.164248110923614090434050 6Q+01

K = 3

0.45050960412736126287316 63Q-01
0.67671679703412590166214 97Q+00
0.30751072425531379720505 34Q+01

K = 4

0.16667473554838058766465 28Q-01
0.31801818427581165376068 90Q+00
0.15524710131218958770477 65Q+01
0.46664433290474544104250 80Q+01

K = 5

0.6575286461353594336586 5520Q-02
0.1606142317401454028672 786Q+00
0.8655532227468952548875 133Q+00
0.2654640475454163235703 710Q+01
0.6359556084008965145949 768Q+01

K = 6

0.27078647018167353281213 92Q-02
0.85083051484286405872436 7Q-01
0.50920574336882041339069 09Q+00
0.16399725587141331793939 92Q+01
0.39152485571787937251731 08Q+01
0.81242826622949267823023 15Q+01

K = 7

0.11497914485285183229071 99Q-02
0.46647839792946826532566 81Q-01
0.31062654534472793642126 58Q+00
0.10582112225813305334731 752Q+01
0.25905418852819393416160 39Q+01
0.52923803397806698510599 54Q+01
0.99425416576636146131905 16Q+01

K = 8

0.49943122984464278450657 26Q-03
0.26249621910425722940990 59Q-01
0.19464308515881936470177 47Q+00
0.70306096717695382871234 12Q+00
0.17791885986441863658766 74Q+01
0.36785450911819020279408 86Q+01
0.67588908712883278280746 05Q+01
0.11802560293662076062522 95Q+02

K = 9

0.22075213177249258589384 88Q-03
0.15075523932511167225267 09Q-01
0.12453931109617932616015 31Q+00
0.47738460783710635550094 626Q+00
0.12525238413602542256205 65Q+01
0.26429408999718080630816 64Q+01
0.48758858205833715552613 97Q+01
0.82962843574703760968473 07Q+01
0.13696204494616620717708 29Q+02

K = 10

0.98923959342615954896268 47Q-04
0.88012649578631899926818 21Q-02
0.81031777614698199447164 06Q-01
0.32975754189759843206795 08Q+00
0.89809311367958967240772 70Q+00
0.19404978203662116331960 74Q+01
0.36249931104443171673481 84Q+01
0.61620580548228508707459 52Q+01
0.98914005871002934495084 36Q+01
0.15617596748110409505771 30Q+02

K = 11

0.44823203426575744895671 33Q-04
0.52079096231622714693951 25Q-02
0.53454753602110594138239 49Q-01
0.23099065581081352737763 85Q+00
0.65336786616145679803640 82Q+00
0.14477018296263370134535 97Q+01
0.27481048927326722791116 29Q+01
0.47058763795251679245770 51Q+01
0.75217998532462429863332 88Q+01
0.11534535332176445137988 17Q+02
0.17562338602158261671094 16Q+02

K = 12

0.20494814735885028389632 31Q-04
0.31164349710881325981811 22Q-02
0.35671293788086112260485 81Q-01
0.16370381064385522438722 50Q+00
0.48102263368462321873361 23Q+00
0.10937497897582915613546 13Q+01
0.21132725514810961116773 49Q+01
0.36586475470203509815728 65Q+01
0.58702581022083758847095 33Q+01
0.89434999106754442116872 19Q+01
0.13218318830687208032368 93Q+02
0.19527044361767893008995 42Q+02

TABLE 4.1b

The weights $A_{k,i}$ for the new numerical quadrature formulas $\int_0^\infty e^{-x} f(x)\, dx \approx \sum_{i=1}^k A_{k,i} f(x_{k,i})$.
The $x_{k,i}$ are given in Table 4.1a.

K = 2

0.4262790219225514332710374Q+00
0.5737209780774485667289626Q+00

K = 3

0.1719726235072209047509659Q+00
0.6479430891839743129379293Q+00
0.1800842873088047823111049Q+00

K = 4

0.7226231915410304004686545Q-01
0.4675117715507984434590670lQ+00
0.4166703847548192995002485Q+00
0.4355558058309322546221596Q-01

K = 5

0.3162835785818733939584087Q-01
0.2964059964296510953358956Q+00
0.4836094609514361588023003Q+00
0.1792932701658073980412668Q+00
0.9062914594918008424696461Q-02

K = 6

0.1428211335995379924913036Q-01
0.1810029734952154472350761Q+00
0.4283183120895699928286281Q+00
0.3139891339858145485368878Q+00
0.6070052264960922735114748Q-01
0.1706944419836984799130211Q-02

K = 7

0.6598330798239452961485086Q-02
0.1098248922206528668845921Q+00
0.3358101635284652760831161Q+00
0.3771311701037623536461123Q+00
0.1528537231889653157956051Q+00
0.1748243331749687194914008Q-01
0.2992868424178626799491478Q-03

K = 8

0.3099870283959833613571391Q-02
0.6693866681287097397256413Q-01
0.2485362003123865502072833Q+00
0.3727396542211409727822777Q+00
0.2431885810309434298663758Q+00
0.6097027090259757966354025Q-01
0.4477052871576393867391677Q-02
0.4970356454266026995976381Q-04

K = 9

0.1474469400701438764603264Q-02
0.4112742988307036543831182Q-01
0.1790200930281806170237407Q+00
0.3306862367379120398001669Q+00
0.2998224185517629141010072Q+00
0.1258263232137970000115962Q+00
0.2098706022864679068141468Q-01
0.1048060132838918194681657Q-02
0.7908823089915984477495928Q-05

K = 10

0.7079022835498523841888713Q-03
0.2548832559511835547461936Q-01
0.1274226523339195729573121Q+00
0.2757347080459756551452594Q+00
0.3172971898747736645572772Q+00
0.1915255678928506327037948Q+00
0.5514876022240948056329139Q-01
0.6444963104831028723924859Q-02
0.2284860115590542510026685Q-03
0.1215522309001340879340377Q-05

K = 11

0.3422920264581562223112103Q-03
0.1592627133604990388353136Q-01
0.9032697757481774988251322Q-01
0.2217346103387849083743623Q+00
0.3052359123969847078321431Q+00
0.2407992784200027993700258Q+00
0.1025645532644478690600888Q+00
0.2121654709984841223948429Q-01
0.1806373139670102934608411Q-02
0.4700289257744581260633884Q-04
0.1815103579438832505612010Q-06

K = 12

0.1664210817022919684179579Q-03
0.1002537753936819105781108Q-01
0.6402700468555379941006606Q-01
0.1745721822360148306828806Q+00
0.2762012799800637007904487Q+00
0.2670162411605854677879026Q+00
0.1524436358317874768111974Q+00
0.4772451419171662163940487Q-01
0.7344423584282093647907940Q-02
0.4696806037099412766549128Q-03
0.9212653029824790631065364Q-05
0.2645218576013667721828686Q-07

TABLE 4.1c

The weights $A_{k,i}$ for the new numerical quadrature formulas $\int_0^\infty E_1(x)f(x)\,dx \approx \sum_{i=1}^k A_{k,i}f(x_{k,i})$. The $x_{k,i}$ are given in Table 4.1a.

K = 2

0.7580234232710699835513690Q+00
0.2419765767289300164486310Q+00

K = 3

0.4554540735108335506704450Q+00
0.4982740688105893984787207Q+00
0.4627185767857705085083424Q-01

K = 4

0.2559124314488475647034520Q+00
0.5540725756664822258223339Q+00
0.1821354548398085887398495Q+00
0.7879538044861620734364579Q-02

K = 5

0.1402073401311772741711620Q+00
0.4891155383497700586633836Q+00
0.3175517522863648430356046Q+00
0.5187500819320468386324305Q-01
0.1250361039483140266607380Q-02

K = 6

0.7590679384919516304074620Q-01
0.3882569073353100374774409Q+00
0.3910982789957450034559817Q+00
0.1317949666122117178283279Q+00
0.1275405893635085165625092Q-01
0.1889942711872265439523386Q-03

K = 7

0.4081165699922440555525246Q-01
0.2916484563421986709994837Q+00
0.4031742897215124443059980Q+00
0.2164039357580742197306810Q+00
0.4509850270751601688195192Q-01
0.2835608913318645418197694Q-02
0.2754955815556210843516167Q-04

K = 8

0.2183694095237601731347362Q-01
0.2123436066706023280859898Q+00
0.3759015998592184903536997Q+00
0.2797841689691778014757056Q+00
0.9604431756368414199484653Q-01
0.1350041606147886266984887Q-01
0.5850466653776554364622037Q-03
0.3903258084702669973656278Q-05

K = 9

0.1163948148700660433391451Q-01
0.1517165526134532491022856Q+00
0.3298341874710617676412120Q+00
0.3135048910910120987495115Q+00
0.1530007866011305814318330Q+00
0.3653427788604604752980879Q-01
0.3655427407496274166283388Q-02
0.1138550570413118897646681Q-03
0.5403857520651553866290788Q-06

K = 10

0.6183607395711354532866023Q-02
0.1071091101886063889549707Q+00
0.2784453407952368949679836Q+00
0.3208016909050212375820361Q+00
0.2033397579357152368218497Q+00
0.7082269642288027252226315Q-01
0.1236216945865997469233175Q-01
0.9144211000825651826936216Q-03
0.2113241163514073803417835Q-04
0.7338645093400497129421534Q-07

K = 11

0.3275384530104277670293281Q-02
0.7501957741413657025016744Q-01
0.2291117291836910307454911Q+00
0.3090943534216158917513319Q+00
0.2398422278869326897396240Q+00
0.1107670005330865349182265Q+00
0.2886106586082509163584454Q-01
0.3810438741749370467894437Q-02
0.2144416921941403642591808Q-03
0.3770931329409887434130059Q-05
0.9804334992569433381436212Q-08

K = 12

0.1730217229647563323727468Q-02
0.5225852598266425798596882Q-01
0.1852238054344086666606745Q+00
0.2857579736265443988113541Q+00
0.2607043547367742704644881Q+00
0.1497818572726624001050412Q+00
0.5279170686778839885617192Q-01
0.1061556042946016328951724Q-01
0.1087699857196495210286065Q-02
0.4764646687998658382098453Q-04
0.6508045093724211033391325Q-06
0.1291464026287846155793969Q-08

TABLE 4.2

S_k and G_k stand for the k-point new rule and Gauss-Laguerre rule respectively, with $w(x) = e^{-x}$. The abscissas for S_k are those given in Table 4.1a, and the corresponding weights are given in Table 4.1b.

Rule	$\|J[f] - J_k[f]\|$ for $J[f] = \int_0^\infty e^{-X}f(x)dx$					
	$f(x) = \dfrac{1}{1+x^2}$	$f(x) = \dfrac{1}{4+x^2}$	$f(x) = \dfrac{1}{e^X+1}$	$f(x) = \dfrac{1}{1+x}$	$f(x) = \dfrac{1}{2+x}$	$f(x) = e^{-x}$
S_2	5×10^{-2}	8×10^{-3}	2×10^{-4}	4×10^{-3}	4×10^{-3}	2×10^{-2}
G_2	3×10^{-2}	6×10^{-3}	3×10^{-3}	3×10^{-2}	4×10^{-3}	2×10^{-2}
S_4	5×10^{-4}	1×10^{-3}	9×10^{-4}	4×10^{-4}	1×10^{-5}	1×10^{-4}
G_4	2×10^{-2}	3×10^{-4}	2×10^{-5}	3×10^{-3}	2×10^{-4}	3×10^{-4}
S_6	2×10^{-3}	1×10^{-4}	1×10^{-4}	2×10^{-5}	2×10^{-6}	1×10^{-5}
G_6	3×10^{-4}	1×10^{-4}	4×10^{-5}	6×10^{-4}	2×10^{-5}	5×10^{-6}
S_8	1×10^{-4}	1×10^{-5}	1×10^{-5}	5×10^{-7}	6×10^{-8}	3×10^{-7}
G_8	1×10^{-3}	1×10^{-5}	4×10^{-6}	1×10^{-4}	3×10^{-6}	7×10^{-8}
S_{10}	8×10^{-5}	1×10^{-6}	9×10^{-7}	2×10^{-8}	2×10^{-10}	3×10^{-8}
G_{10}	6×10^{-4}	8×10^{-6}	1×10^{-6}	4×10^{-5}	4×10^{-7}	1×10^{-9}
S_{12}	2×10^{-6}	5×10^{-7}	9×10^{-9}	3×10^{-9}	1×10^{-10}	3×10^{-10}
G_{12}	1×10^{-4}	8×10^{-7}	2×10^{-7}	1×10^{-5}	1×10^{-7}	1×10^{-11}
Exact	$0.62144\ldots$	$0.19951\ldots$	$0.30685\ldots$	$0.59634\ldots$	$0.36132..$	0.5

The abscissas $x_{k,i}$ for the weight functions $w_1(x) = x^\alpha e^{-x}$ and $w_2(x) = x^\alpha E_p(x)$, $\alpha > -1$, $p + \alpha > 0$, have been determined for different values of α and those for $\alpha = 0$, $k = 2(1)12$, are given in Table 4.1a. Recall that for a fixed value of α, $w_1(x)$ and $w_2(x)$ (for all p such that $\alpha + p > 0$) have the same set of abscissas obtained by solving the polynomial equation $\sum_{j=0}^k \lambda_j z^j = 0$, where

$$(4.1) \qquad \lambda_j = (-1)^j \binom{k}{j} \frac{(j+1)^k}{(\alpha + j)!}, \qquad j = 0, 1, \ldots, k.$$

Once the abscissas $x_{k,i}$ have been computed, the corresponding weights $A_{k,i}$ can be determined from the formula

$$(4.2) \qquad A_{k,i} = \left. \frac{\sum_{j=0}^k \lambda_j z^j A_j}{\sum_{j=0}^k j \lambda_j z^{j-1}} \right|_{z = x_{k,i}}, \qquad i = 1, \ldots, k.$$

In Table 4.2 we compare the approximations $J_k[f]$ obtained by using the new numerical quadrature formulas and the corresponding Gauss-Laguerre rules with $w(x) = e^{-x}$. The abscissas for the new rules are those given in Table 4.1a, and the

TABLE 4.3

S_k and G_k stand for the k-point new rule and Gauss-Laguerre rule respectively with $w(x) = e^{-x}$, i.e., the singularities of the integrands $f(x)$ at $x = 0$ are ignored. The abscissas for S_k are those given in Table 4.1a, and the corresponding weights are given in Table 4.1b.

Rule	$|J[f] - J_k[f]|$ for $J[f] = \int_0^\infty e^{-x}f(x)dx$				
	$f(x) = x^{-\frac{1}{2}}$	$f(x) = \log x$	$f(x) = x^{\frac{1}{2}}\log x$	$f(x) = x^{\frac{1}{2}}$	$f(x) = x^{3/2}$
S_2	2×10^{-1}	9×10^{-3}	2×10^{-2}	6×10^{-3}	1×10^{-1}
G_2	6×10^{-1}	3×10^{-1}	5×10^{-2}	4×10^{-2}	2×10^{-2}
S_4	3×10^{-2}	4×10^{-3}	7×10^{-4}	1×10^{-6}	3×10^{-4}
G_4	4×10^{-1}	2×10^{-1}	3×10^{-2}	1×10^{-2}	3×10^{-3}
S_6	8×10^{-4}	6×10^{-4}	8×10^{-4}	3×10^{-5}	2×10^{-7}
G_6	3×10^{-1}	1×10^{-1}	2×10^{-2}	7×10^{-3}	1×10^{-3}
S_8	3×10^{-3}	1×10^{-4}	3×10^{-5}	3×10^{-6}	8×10^{-8}
G_8	3×10^{-1}	8×10^{-2}	1×10^{-2}	5×10^{-3}	5×10^{-4}
S_{10}	2×10^{-3}	1×10^{-4}	7×10^{-7}	9×10^{-7}	3×10^{-8}
G_{10}	3×10^{-1}	6×10^{-2}	1×10^{-2}	3×10^{-3}	3×10^{-4}
S_{12}	1×10^{-3}	2×10^{-5}	3×10^{-6}	4×10^{-7}	4×10^{-9}
G_{12}	3×10^{-1}	5×10^{-2}	8×10^{-3}	3×10^{-3}	2×10^{-4}
G_{16}	2×10^{-1}	4×10^{-2}	6×10^{-3}	2×10^{-3}	6×10^{-5}
G_{24}	2×10^{-1}	3×10^{-2}	3×10^{-3}	9×10^{-4}	3×10^{-5}
G_{32}	2×10^{-1}	2×10^{-2}	2×10^{-3}	6×10^{-4}	1×10^{-5}
Exact	$1.7724...$	$-0.57721...$	$0.032338...$	$0.88622...$	$1.3293...$

corresponding weights are given in Table 4.1b. As can be seen from Table 4.2, for analytic $f(x)$, the new rules compare favorably with the Gaussian rules and in some cases give better results. It is worth noting that, as the singularities of $f(x)$ become farther away from the interval of integration $[0, \infty)$, the $J_k[f]$ become better for both rules.

Since the abscissas of the new rules that are given in Table 4.1 do not include the lower limit $x = 0$ (a property shared by the Gaussian rules too), we can use the new rules for obtaining approximations to integrals for which $f(x)$ are singular at

TABLE 4.4

S_k and G_k stand for the k-point new rule and Gaussian rule respectively with $w(x) = E_1(x)$. The abscissas for S_k are those given in Table 4.1a, and the corresponding weights are given in Table 4.1c. The abscissas and weights for G_{10} have been taken from the 14-digit tables of Danloy [4].

Rule	$|J[f] - J_k[f]|$ for $J[f] = \int_0^\infty E_1(x)f(x)dx$					
	$f(x) = \dfrac{1}{1+x^2}$	$f(x) = \dfrac{1}{4+x^2}$	$f(x) = \dfrac{1}{e^x+1}$	$f(x) = \dfrac{1}{1+x}$	$f(x) = \dfrac{1}{2+x}$	$f(x) = e^{-x}$
S_2	2×10^{-3}	3×10^{-3}	4×10^{-4}	1×10^{-2}	3×10^{-3}	2×10^{-2}
S_4	5×10^{-3}	3×10^{-4}	8×10^{-5}	2×10^{-4}	2×10^{-5}	3×10^{-4}
S_6	1×10^{-3}	1×10^{-5}	1×10^{-5}	9×10^{-7}	2×10^{-6}	4×10^{-6}
S_8	2×10^{-4}	4×10^{-6}	2×10^{-7}	1×10^{-6}	3×10^{-8}	2×10^{-8}
S_{10}	3×10^{-5}	1×10^{-6}	2×10^{-7}	1×10^{-7}	1×10^{-9}	2×10^{-10}
G_{10}	1×10^{-4}	2×10^{-6}	6×10^{-9}	2×10^{-5}	2×10^{-7}	2×10^{-10}
S_{12}	2×10^{-5}	1×10^{-8}	6×10^{-8}	1×10^{-8}	2×10^{-10}	1×10^{-12}
Exact	$0.80759\ldots$	$0.22806\ldots$	$0.39225\ldots$	$0.74519\ldots$	$0.41835\ldots$	$0.69314\ldots$

$x = 0$, by avoiding the singularity. It turns out that the new formulas for $w(x) = e^{-x}$ work very efficiently on integrals of the form $\int_0^\infty e^{-x}f(x)\,dx$, where the functions $f(x)$ have algebraic or logarithmic singularities or products of them at $x = 0$, especially when $f(x)$ are continuous at $x = 0$. For such integrals, numerous computations have shown that the new rules are superior to the Gaussian rules with $w(x) = e^{-x}$. In Table 4.3 we compare the new rules and the Gaussian rules, with $w(x) = e^{-x}$, for a set of singular functions. We note that as we proceed to the right along this table, the $f(x)$ become less and less singular.

In Table 4.4 and Table 4.5 we compare the new rules and the Gaussian rules with $w(x) = E_1(x)$, on the same functions $f(x)$ that appear in Table 4.2 and Table 4.3 respectively. The abscissas for the new rules are again those given in Table 4.1a, and the corresponding weights are given in Table 4.1c. The results for the Gaussian rules in this case have been obtained by using the 10-point and 20-point 14-digit tables of Danloy [4]. The conclusions from this comparison are the same as those for the case $w(x) = e^{-x}$.

S_k and G_k stand for the k-point new rule and Gaussian rule respectively with $w(x) = E_1(x)$; i.e., the singularities of the integrands $f(x)$ at $x = 0$ are ignored. The abscissas for S_k are those given in Table 4.1a, and the corresponding weights are given in Table 4.1c. The abscissas and weights for G_{10} and G_{20} have been taken from the 14-digit tables of Danloy [4].

Rule	$|J[f] - J_k[f]|$ for $J[f] = \int_0^\infty E_1(x)f(x)dx$				
	$f(x) = x^{-\frac{1}{2}}$	$f(x) = \log x$	$f(x) = x^{\frac{1}{2}}\log x$	$f(x) = x^{\frac{1}{2}}$	$f(x) = x^{3/2}$
S_2	1×10^0	2×10^{-1}	3×10^{-2}	2×10^{-3}	2×10^{-2}
S_4	4×10^{-1}	1×10^{-2}	5×10^{-3}	2×10^{-3}	7×10^{-5}
S_6	1×10^{-1}	1×10^{-2}	6×10^{-4}	2×10^{-5}	4×10^{-6}
S_8	8×10^{-3}	1×10^{-3}	2×10^{-4}	5×10^{-5}	1×10^{-6}
S_{10}	3×10^{-2}	4×10^{-4}	8×10^{-5}	4×10^{-6}	4×10^{-8}
G_{10}	1×10^0	2×10^{-1}	3×10^{-2}	7×10^{-3}	4×10^{-4}
S_{12}	2×10^{-2}	3×10^{-4}	5×10^{-6}	3×10^{-6}	4×10^{-8}
G_{20}	1×10^0	1×10^{-1}	1×10^{-2}	3×10^{-3}	8×10^{-5}
Exact	$3.5449\ldots$	$-1.5772\ldots$	$-0.37231\ldots$	$0.59081\ldots$	$0.53173\ldots$

5. Extension to Doubly Infinite Range Integrals.

In this section we deal with integrals of the form

$$(5.1) \qquad J[f] = \int_{-\infty}^{+\infty} w(x)f(x)\, dx,$$

where $w(x)$ is an even function of x and all its moments exist. Then

$$(5.2) \quad H(z) = \int_{-\infty}^{+\infty} \frac{w(x)}{z - x}\, dx = 2z \int_0^\infty \frac{w(x)}{z^2 - x^2}\, dx \sim \sum_{i=1}^\infty \frac{\nu_i}{z^{2i-1}} \quad \text{as } z \to \infty,$$

where

$$(5.3) \qquad \nu_i = 2\int_0^\infty w(x)x^{2i-2}\, dx, \qquad i = 1, 2, \ldots.$$

If we let $w(x) = |x|^\beta e^{-x^2}$ and make the change of variable $x^2 = t$ in (5.2), then

$$(5.4) \qquad H(z) = z\int_0^\infty \frac{t^{(\beta-1)/2}e^{-t}}{z^2 - t}\, dt$$

and

$$(5.5) \qquad \nu_i = \left[\left(\frac{\beta - 1}{2} \right) + i - 1 \right]!, \qquad i = 1, 2, \dots.$$

This $H(z)$ is the same as that for $w(x) = x^\alpha e^{-x}$ dealt with in Sections 2, 3 and 4, with $\alpha = (\beta - 1)/2$ and z replaced by z^2. If we let $A_0 = 0$, $A_r = \sum_{i=1}^r \nu_i / z^{2i-1}$, $r = 1, 2, \dots$, and $R_r = \nu_r z^{-2r+1} r^{-1}$, $r = 1, 2, \dots$, then the approximations $T_{k,n}$ to $H(z)$ are again very accurate. Of course, $T_{k,n}$ also in this case is a rational function of z. $T_{k,n}$ has simple poles only when $n = 1, 2$. When $n = 1$, the numerator of $T_{k,n}$ is an odd polynomial of degree $\leqslant 2k - 1$ and its denominator is an even polynomial of degree $2k$. The zeros of the denominator are $\pm \sqrt{\xi_i}$, $i = 1, 2, \dots, k$, where ξ_i are the zeros of $((d/d\xi)\xi)^k L_k^{(s)}(\xi)$, $s = (\beta - 1)/2$. When $n = 2$ the numerator of $T_{k,2}$ is an even polynomial of degree $\leqslant 2k$ and its denominator is an odd polynomial of degree $2k + 1$. In this case the zeros of the denominator are 0 and $\pm \sqrt{\xi_i}$, $i = 1, 2, \dots, k$, where ξ_i are the positive zeros of $((d/d\xi)\xi)^k [L_k^{(s)}(\xi)]$, $s = (\beta - 1)/2 + 1$. Therefore, the rational approximations $T_{k,1}$ and $T_{k,2}$ give rise to numerical quadrature formulas with $2k$ and $2k + 1$ abscissas, respectively. For these formulas the weights corresponding to the nonzero abscissas $\pm \sqrt{\xi_i}$ are equal. Numerical computations with these formulas show that they are practically as efficient as the corresponding Gaussian formulas.

6. Concluding Remarks. In this work we have used the approach of Sidi [8] to obtain new numerical quadrature formulas for infinite range integrals of the form $\int_0^\infty w(x)f(x)\, dx$, where $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, or $w(x) = x^\alpha E_p(x)$, $\alpha > -1$, $\alpha + p > 0$. We have shown that the abscissas of these formulas are related to the generalized Laguerre polynomials and furthermore we have proved that they are distinct and lie in the interval of integration. The weights of these formulas turn out to be all positive although no proof of this is available yet. We have shown that these formulas are of interpolatory type. An important advantage of these formulas is that for a fixed value of α, both $w(x) = x^\alpha e^{-x}$ and $w(x) = x^\alpha E_p(x)$ have the same set of abscissas independent of p. Finally, comparison of the new formulas with the Gaussian formulas indicate that the former, on the average, are better than the latter.

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